

Convex Domains of Finite Type

JEFFERY D. MCNEAL*

*Department of Mathematics, Princeton University,
Princeton, New Jersey 08544*

Communicated by R. B. Melrose

Received November 20, 1991

We construct bounded, plurisubharmonic functions with maximally large Hessians near the boundary of a smoothly bounded convex domain in \mathbb{C}^n . As a corollary, the equality of the order of contact of the boundary with complex analytic varieties (the D'Angelo type) and the order of contact with complex lines is demonstrated. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let $\Omega \subset \subset \mathbb{C}^n$ be smoothly bounded and pseudoconvex and let $p \in b\Omega$ be some point in the boundary of Ω near which we seek to examine the behavior of holomorphic functions on Ω . If the Levi form vanishes at p , it has been known for some time that many forms of this behavior are controlled by higher than second-order invariants of $b\Omega$ related to the order of vanishing of the Levi form. In particular, the *type* of p , as defined by D'Angelo, has been shown by Catlin to be the quantity which determines whether there is a subelliptic estimate for the $\bar{\partial}$ -Neumann problem in a neighborhood of p and, if there is, how strong an estimate exists. This notion of type (sometimes with additional hypotheses) has also been shown to imply the existence of various kinds of peaking functions, control the boundary behavior of many domain dependent, canonical kernels, and generate holomorphic functions with Lebesgue class growth. The literature on these question is very extensive; we refer the reader to the bibliography in [F–K] and the upcoming book of D'Angelo as guides to some of these results.

Computing the type of a boundary point, or even determining whether a given point is of finite type, is not always a simple matter. In principle, the work of D'Angelo shows how to decide this question for a given domain by using a well-defined algorithm on a class of ideals associated to

* Research partially supported by an NSF postdoctoral fellowship.

the local defining function of the boundary. The ideals, however, are often quite large and the required computations with them fairly complicated. Some of the complications arise from the necessity of considering singular complex analytic varieties during this process. As an illustration, using these principles to compute the type of 0, in the boundary of the domain defined by

$$r(z) = 2 \operatorname{Re} z_3 + |z_1^2 - z_2^3|^2 + |z_1|^8 + |z_1|^{18} - |z_2|^{12},$$

we must consider the ideal generated by the functions

$$z_3, \quad z_1^2 - z_2^3 - \alpha z_2^6, \quad z_1^4 - \alpha z_2^6, \quad z_1^9 - \alpha z_2^6,$$

where α is any complex number. Only after considerable work can it be shown that the type of 0 is 27.

One of the purposes of this paper is to show that, if Ω is convex near p , the situation is greatly simplified and the type of p can be computed by only considering the order of contact of $b\Omega$, at p , with complex lines.

THEOREM 1.1. *Suppose that $\Omega \subset \subset \mathbb{C}^n$ is a smoothly bounded domain, $p \in b\Omega$, and Ω is convex near p . If the line type of p is $L < \infty$, then the variety type of p , denoted by $\Delta_1(p)$, is also finite and $L = \Delta_1(p)$.*

The proof we give of this result is analytic in character and relies in an essential way on a construction of certain plurisubharmonic functions on convex domains. The proof also uses a theorem of Catlin on fitting manifolds of maximum diameter inside domains near a boundary point of finite type. The plurisubharmonic functions we construct have intrinsic interest and are the main point of this paper. In a future paper, we will show how they may be used to prove that the sharp subelliptic estimate for the $\bar{\partial}$ -Neumann problem on $(0, 1)$ -forms holds near p , a fact previously established by Fornæss and Sibony in [F-S]. These functions are also an essential ingredient in our upcoming description of the boundary behavior of the Bergman kernels of convex domains.

The organization of this paper is as follows. In Section 2 we define the relevant notions we use and recall the aforementioned result of Catlin. In Section 3, we construct plurisubharmonic functions on certain polydiscs which have maximally large Hessians, subject to the restriction that these functions are bounded. The proof of Theorem 1.1 is given in Section 4.

Remarks. 1. J.-H. Chen, in his Purdue University dissertation [Ch], has discovered the same orthogonalization procedure which gives the coordinates in Section 3. He also constructs plurisubharmonic functions very similar to ours and uses them to estimate the differential metrics of Caratheodory, Bergman, and Kobayashi on convex domains.

2. J. E. Fornæss [F] has also observed that the conclusion of Theorem 1.1 holds. He obtains the result after using the methods in the above mentioned [F-S] for constructing Hölder continuous plurisubharmonic functions on convex domains.

3. An important open problem is to characterize the class of domains for which it suffices to compute the order of contact of the boundary with complex manifolds in order to measure the variety type. This is not true for general smooth pseudoconvex domains as, for instance, the example

$$\Omega = \{z : r(z) = 2 \operatorname{Re} z_3 + |z_1^2 - z_2^2|^2 < 0\}, \quad p = 0$$

shows. A trivial corollary of Theorem 1.1 is that (biholomorphic images of) convex domains belong to this class.

I would like to thank D. W. Catlin for informing me about Chen's work and for several suggestions which improved the exposition of this paper. I also thank I. Graham and M. Range for their careful readings of a draft of this paper.

2. DEFINITIONS AND CATLIN'S THEOREM

Let $\Omega \subset \subset \mathbb{C}^n$ be a domain with smooth boundary and let $p \in b\Omega$. For a neighborhood U of p , fix a smooth real-valued function r so that

$$\Omega \cap U = \{z \in U : r(z) < 0\}$$

and $\nabla r \neq 0$ on $b\Omega \cap U$. If (z_1, \dots, z_n) are the standard coordinates on \mathbb{C}^n and we denote $z_i = x_i + ix_{i+n}$, then Ω is convex in U if

$$\sum_{i,j=1}^{2n} \frac{\partial^2 r}{\partial x_i \partial x_j} (p) t_i t_j \geq 0 \quad \text{whenever} \quad \sum_{i=1}^{2n} \frac{\partial r}{\partial x_i} (p) t_i = 0$$

for all $p \in b\Omega \cap U$ and $t \in \mathbb{R}^{2n}$.

If f is a smooth, complex-valued function, defined near the origin in \mathbb{C} , let $v(f)$ denote the order of vanishing of $f - f(0)$ at the origin. For a vector-valued $F = (f_1, \dots, f_n)$, let $v(F) = \min_i v(f_i)$. The following definition was formulated by D'Angelo in [D'A].

DEFINITION 1. p is a point of finite (one-dimensional) variety type if there exists a constant m such that

$$A_1(p) = \sup_F \frac{v(r \circ F)}{v(F)} \leq m$$

for F a holomorphic parameterization of a one-dimensional complex analytic subvariety of \mathbb{C}^n with $F(0) = p$. $\Delta_1(p)$ is called the (one-dimensional) variety type of p .

If V is a particular complex analytic subvariety of \mathbb{C}^n passing through p then, in view of Definition 1, we will say that V has order of contact M with $b\Omega$ at p if M is the largest number such that for some $C > 0$,

$$|r(z)| \leq C |z|^M$$

for all $z \in V$, z sufficiently close to 0.

When $b\Omega$ allows a complex variety with high order of contact at $p \in b\Omega$, Catlin [C] showed that there are complex manifolds in Ω near p , with large diameters related to this order of contact.

THEOREM 2.1. (Catlin). *Let Ω be a domain in \mathbb{C}^n with C^1 -boundary and suppose that there is a one-dimensional complex analytic variety with order of contact M with $b\Omega$ at $p \in b\Omega$. Then, in any neighborhood U of p , there is a family of one-dimensional complex manifolds M_t , for $t \in \mathbb{R}^+$, $t \rightarrow 0$, contained in U such that*

(i) M_t is the image of a holomorphic map $g'_t: B(0, t) \rightarrow \mathbb{C}^n$, where $B(0, t)$ is the disc in \mathbb{C} centered at the origin of radius t .

(ii) There is a constant K , independent of t , so that $|dg'_t(z)| \leq K$ for all $z \in B(0, t)$.

(iii) At least one of the components of g'_t vanishes to first order, independent of t , at the origin.

(iv) There exists a constant C , independent of t , so that

$$|r(z)| \leq Ct^M$$

if $z \in M_t$.

A complex line in \mathbb{C}^n is a set of points of the form $\{a\zeta + b : \zeta \in \mathbb{C}\}$ for fixed $a, b \in \mathbb{C}^n$. In a manner analogous to Definition 1, we will consider the order of contact of $b\Omega$ with complex lines.

DEFINITION 2. p is a point of finite line type if there exists a constant L such that

$$\sup_l v(r \circ l) \leq L$$

for l a parameterization of a complex line with $l(0) = p$. The smallest L for which the inequality holds will be called the line type of 0.

Finally, we recall some standard notation which will simplify writing the results in the next section. For $z \in \mathbb{C}^n$, let $\delta(z)$ denote the distance from z to $b\Omega$. If A and B are functions depending on several parameters, we will write $A \lesssim B$ or $B \gtrsim A$ to mean that there is a constant C , independent of a certain number of the parameters, such that $|A| \leq C|B|$. The particular parameters of which the constant is independent will be specified or clear in context.

3. CONSTRUCTION OF PLURISUBHARMONIC FUNCTIONS

Before constructing the plurisubharmonic functions we will need to prove Theorem 1.1, we introduce some useful coordinates, centered at an arbitrary point near the p , which reflect the shape of $b\Omega$ near this point. In all the following, \lesssim means the constants involved are independent of z' and δ .

PROPOSITION 3.1. *Let $\Omega \subset \subset \mathbb{C}^n$ be smoothly bounded with $p \in b\Omega$. Suppose that U is neighborhood of p in which Ω is convex. Assume, also, that the line type of p is $L < \infty$.*

After perhaps shrinking U , for every $z' \in \Omega \cap U$ with $\delta(z') = \delta$, there exist coordinates (z_1, \dots, z_n) centered at z' , positive numbers τ_1, \dots, τ_n with $\tau_1 = \delta$, and points $p_1, \dots, p_n \in b\Omega$ such that, in the coordinates (z_1, \dots, z_n) , the defining function r satisfies

(i) if $i > j$,

$$\left| \frac{\partial r}{\partial z_i}(p_j) \right| = 0;$$

(ii) for $1 \leq i \leq n$,

$$\frac{\delta}{\tau_i} \lesssim \left| \frac{\partial r}{\partial z_i}(p_i) \right| \lesssim \frac{\delta}{\tau_i};$$

(iii) if $i < j$,

$$\left| \frac{\partial r}{\partial z_i}(p_j) \right| \lesssim \frac{\delta}{\tau_i}.$$

Also, if we define the polydisc

$$P_\delta(z') = \{z \in U : |z_1| < \tau_1, \dots, |z_n| < \tau_n\},$$

then there exists a constant $C > 0$, independent of $z' \in \Omega \cap U$, such that $CP_\delta(z') \subset \Omega$.

Proof. For U small enough and $z' \in \Omega \cap U$, there is a unique nearest point in $b\Omega$ to z' . Call this point p_1 and let z_1 be a parameterization of the complex line from z' to p_1 with $z_1(0) = z'$ and p_1 lying on the positive $\operatorname{Re} z_1$ axis. The number τ_1 is defined to be $\delta(z')$. Now consider the distance from z' to $b\Omega$ along any complex line orthogonal to the vector from z' to p_1 ; that is, any complex line such that all real vectors contained in it are perpendicular to this vector. The largest such distance is $\lesssim \delta^{1/L}$ by our hypothesis on the line type of 0. Let τ_2 be this maximum distance and let $p_2 \in b\Omega$ be any point at which this distance is realized. Choose a parameterization of the line from z' to p_2 whose value at 0 is z' and so that p_2 lies on the positive real axis of the complex parameter; call this parameter the coordinate z_2 . We now consider the orthogonal complement of the subspace spanned by the vector from z' to p_1 and the vector from z' to p_2 . Solving the extreme value problem as in the above procedure, we find the number τ_3 , a point $p_3 \in b\Omega$, and the coordinate function z_3 which appropriately parameterizes the line from z' to p_3 . Inductively, we continue this process and obtain the n coordinate functions z_i with the weights τ_i and the distinguished points p_i .

The construction of the coordinates (z_1, \dots, z_n) and the fact that the points p_i solve the constrained maximum problem show that (i) holds. Also, since each $p_i \in b\Omega \cap U$ and Ω is convex in U , we may take $C = 1/4^n$ and see immediately that $CP_\delta(z') \subset \Omega$.

The verification of (ii) and (iii) for $i = 1$ is also immediate. In fact, since $(\partial r / \partial z_1)(0) \neq 0$, we have

$$1 \lesssim \left| \frac{\partial r}{\partial z_1}(z) \right| \lesssim 1 \quad (3.1)$$

for all $z \in U \cap \Omega$, in particular for $z = p_i$, $1 \leq i \leq n$. Equation (3.1) also holds with $\partial r / \partial x_1$ in place of $\partial r / \partial z_1$, where $z_1 = x_1 + iy_1$, by our coordinate construction. To see (ii) and (iii) in general, note that the tangent plane to $b\Omega$ at each p_i is described by the equation

$$\operatorname{Re}[\partial r(p_i) \cdot (z - p_i)] = 0, \quad (3.2)$$

where $\partial r(p_i)$ denotes the complex gradient of r evaluated at p_i . Setting $z_2 = \dots = z_i = 0$ and using (i) gives

$$\operatorname{Re} \left[z_1 \frac{\partial r}{\partial z_1}(p_i) - \tau_i \frac{\partial r}{\partial z_i}(p_i) \right] = 0. \quad (3.3)$$

By convexity, if z_1 lies on the line described by (3.3), then $x_1 \geq \delta$. Thus, (3.1) and (3.3) imply that

$$\left| \operatorname{Re} \tau_i \frac{\partial r}{\partial z_i}(p_i) \right| \geq K\delta$$

or

$$\frac{\delta}{\tau_i} \lesssim \left| \frac{\partial r}{\partial z_i}(p_i) \right|. \quad (3.4)$$

To verify the other inequality in (ii), it is convenient to estimate the numbers τ_i in a more quantitative form. Let N_i be a positive integer, to be chosen in a moment. Note that, for each $2 \leq i \leq n$, Taylor's theorem gives

$$r(0, \dots, z_i, \dots, 0) = r(z') + 2 \operatorname{Re} \sum_{k=2}^{N_i} a_k^i(z') z_i^k + \sum_{l,j=2}^{N_i} b_{lj}^i(z') z_i^l \bar{z}_i^j + \mathcal{O}(|z_i|^{N_i+1})$$

and, from the definition of τ_i , we have that

$$\delta = 2 \operatorname{Re} \sum_{k=2}^{N_i} a_k^i(z') \tau_i^k + \sum_{l,j=2}^{N_i} b_{lj}^i(z') \tau_i^{l+j} + \mathcal{O}(|\tau_i|^{N_i+1}), \quad (3.5)$$

where $\delta = |r(z')|$. Set

$$A_k^i(z') = \max \{ |a_k^i(z')|, |b_{lj}^i(z')| : l+j=k \} \quad (3.6)$$

and for $\delta > 0$,

$$\sigma_i(z', \delta) = \min \{ (\delta/A_k^i(z'))^{1/k} : 2 \leq k \leq N_i \}. \quad (3.7)$$

Our assumption that p has finite line type implies that for each $2 \leq i \leq n$, there exists an integer $N_i \leq L$ and integers ν_i, μ_i , with $\nu_i + \mu_i = N_i$, such that

$$\frac{\partial^{N_i}}{\partial z_i^{\nu_i} \partial \bar{z}_i^{\mu_i}} r(p) \neq 0.$$

By considering only $z' \in U$, for U a small enough neighborhood of p , we thus have

$$\sigma_i(z', \delta) \lesssim \delta^{1/N_i}. \quad (3.8)$$

Let N_i denote such a number, both in the notation above and in the sequel.

We claim that there is a constant C , independent of $z' \in U$, so that, if $\delta = |r(z')|$, we have for each $1 \leq i \leq n$,

$$C^{-1} \sigma_i(z', \delta) \leq \tau_i(z', \delta) \leq C \sigma_i(z', \delta). \quad (3.9)$$

Here we have written $\tau_i(z', \delta)$ for τ_i to emphasize the dependence on z' . To

see (3.9), replace τ_i by σ_i in the right hand side of (3.5) and use (3.7) to obtain

$$2 \operatorname{Re} \sum_{k=2}^{N_i} a_k^i(z') \sigma_i^k + \sum_{l,j=2}^{N_i} b_{lj}^i(z') \sigma_i^{l+j} + \mathcal{O}(\sigma_i^{N_i+1}) \lesssim \delta. \quad (3.10)$$

On the other hand, for each i define

$$\kappa_i = \min\{k : (\delta/A_k^i(z'))^{1/k} = \sigma_i(z', \delta)\}.$$

This defines the approximate δ -type of z' along each of the coordinate axes, very much in the spirit of previous such notions in \mathbb{C}^2 . Set

$$f_i(z_i) = 2 \operatorname{Re} \sum_{k=2}^{N_i} a_k^i(z') z_i^k + \sum_{l,j=2}^{N_i} b_{lj}^i(z') z_i^l \bar{z}_i^j + \mathcal{O}(|z_i|^{N_i+1}),$$

where the \mathcal{O} -term is the same as in (3.5). It follows from the definition of κ_i and the convexity of f_i that

$$\begin{aligned} |f_i(\sigma_i(z', \delta))| &\geq c \left(\sum_{k=2}^{N_i} |a_k^i(z')| \sigma_i^k + \sum_{l,j=2}^{N_i} |b_{lj}^i(z')| \sigma_i^{l+j} \right) + \mathcal{O}(\sigma_i^{N_i+1}) \\ &\geq c \left(\sum_{k=\kappa_i}^{N_i} |a_k^i(z')| \sigma_i^k \right) \\ &\geq c\delta, \end{aligned}$$

which, with (3.10), proves the claim.

It follows directly from (3.9) that, for each $2 \leq i \leq n$,

$$\left| \frac{\partial^k}{\partial z_i^k} r(0) \right| \lesssim \delta \tau_i(z', \delta)^{-k}, \quad k = 2, \dots, N_i.$$

Taylor's theorem then implies that

$$\left| \frac{\partial}{\partial z_i} r(0, \dots, z_i, \dots, 0) \right| \lesssim \sum_{m=0}^{N_i} \delta \tau_i(z', \delta)^{-m-1} |z_i|^m + |z_i|^{N_i+1}.$$

Substituting τ_i for z_i and recalling that $p_i = (0, \dots, \tau_i, \dots, 0)$, we obtain

$$\begin{aligned} \left| \frac{\partial r}{\partial z_i}(p_i) \right| &\lesssim \frac{\delta}{\tau_i} + \tau_i^{N_i+1} \\ &\lesssim \frac{\delta}{\tau_i}, \end{aligned} \quad (3.11)$$

which is the other half of the inequality in (ii).

For (iii), return to (3.2) and set $z_k = 0$ except for $k = j$ to obtain

$$\operatorname{Re} \left[z_j \frac{\partial r}{\partial z_j} (p_i) - \tau_i \frac{\partial r}{\partial z_i} (p_i) \right] = 0.$$

Convexity implies that the line determined by this equation lies completely outside of Ω . Therefore, the distance from z' (the origin in the z -coordinates) is greater than τ_j . Using the upper bound in (ii), we thus have

$$\begin{aligned} \tau_j &\leq \left| \operatorname{Re} \tau_i \frac{\partial r}{\partial z_i} (p_i) \right| \left/ \left| \frac{\partial r}{\partial z_j} (p_i) \right| \right. \\ &\lesssim \delta \left/ \left| \frac{\partial r}{\partial z_j} (p_i) \right| \right|. \end{aligned}$$

This is inequality (iii), after transposition of terms, which finishes the proof of the proposition.

We now want to use the coordinates of Proposition 3.1 to construct plurisubharmonic functions on Ω with two important properties: uniform boundedness on Ω and maximally large Hessian on the polydiscs $P_\delta(z')$, subject to the previous constraint. The convexity of Ω near p suggests the use of the supporting hyperplanes to $b\Omega$ as an aid for constructing plurisubharmonic functions with the boundedness property. On the other hand, the extreme nature of the points p_i shows us how to situate these planes to obtain a function with maximal complex Hessian.

PROPOSITION 3.2. *Let Ω satisfy the hypotheses of Proposition 3.1. Also, let $P_\delta(z')$ and $\tau_2(z', \delta)$ be as in Proposition 3.1. For each $z' \in \Omega \cap U$ there exists a function $\phi \in C^\infty(\Omega)$ satisfying the following properties:*

- (i) $|\phi(z)| \leq 1, z \in \Omega \cap U$;
- (ii) if $z \in P_\delta(z'), \xi \in \mathbb{C}^n$, then

$$\sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} (z) \xi_i \bar{\xi}_j \gtrsim \tau_2(z', \delta)^{-2} \|\xi\|^2;$$

- (iii) ϕ is plurisubharmonic on Ω .

Proof. Fix $z' \in \Omega \cap U$ and, as before, let $\delta = \delta(z')$. For each $k, 1 \leq k \leq n$, set

$$\psi_k(z) = \frac{1}{\delta} (z - p_k) \cdot (\partial r(p_k)).$$

The function ϕ is defined

$$\phi(z) = \frac{1}{n} \sum_{k=1}^n |e^{\psi_k(z)}|^2.$$

Since ϕ is the sum of the moduli squared of holomorphic functions, it is plurisubharmonic on \mathbb{C}^n ; thus (iii) holds. From the fact that Ω is convex in U it follows, after perhaps adjusting the sign of the real coordinates, that

$$\operatorname{Re} \psi_k(z) \leq 0, \quad \text{if } z \in \Omega \cap U,$$

and, so (i) also holds.

To show that ϕ satisfies (ii), simply compute: for $\eta \in \mathbb{C}^n$,

$$\begin{aligned} \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \eta_i \bar{\eta}_j &= \frac{1}{n} \sum_{k=1}^n \left| \sum_{i=1}^n \frac{\partial}{\partial z_i} e^{\psi_k} \eta_i \right|^2 \\ &= \frac{1}{n} \sum_{k=1}^n |e^{\psi_k(z)}|^2 \left| \sum_{i=1}^n \frac{\partial \psi_k}{\partial z_i} \eta_i \right|^2 \\ &= \frac{1}{n} \delta^{-2} \sum_{k=1}^n |e^{\psi_k(z)}|^2 \left| \sum_{i=1}^n \frac{\partial r}{\partial z_i}(p_k) \eta_i \right|^2. \end{aligned} \quad (3.12)$$

If $z \in P_\delta(z')$, then for all $1 \leq k \leq n$, $|\operatorname{Re} \psi_k(z)| \leq 1$. Therefore, (3.12) implies

$$\sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \eta_i \bar{\eta}_j \gtrsim \delta^{-2} \sum_{k=1}^n \left| \sum_{i=1}^n \frac{\partial r}{\partial z_i}(p_k) \eta_i \right|^2 \quad z \in P_\delta(z'). \quad (3.13)$$

Consider the matrix $M = ((\partial r / \partial z_i)(p_k))_{i,k}$. Property (i) in Proposition 3.1 says that this matrix is lower triangular while property (ii) of the same proposition allows us to read off the approximate value of its diagonal entries. There is, therefore, a unitary matrix \mathcal{U} such that

$$\mathcal{U}M = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix},$$

where

$$\frac{\delta}{\tau_i(z', \delta)} \lesssim \lambda_i \lesssim \frac{\delta}{\tau_i(z', \delta)}.$$

Note that the right-hand side of (3.13) is $\delta^{-2} \|\mathcal{U}M \cdot \eta\|^2$, which is the same as $\delta^{-2} \|\mathcal{U}M \cdot \eta\|^2$, since \mathcal{U} is unitary. Thus, (3.13), implies

$$\begin{aligned} \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} (z) \eta_i \bar{\eta}_j &\gtrsim \delta^{-2} \sum_{k=1}^n |\lambda_k|^2 |\eta_k|^2 \\ &\geq \delta^{-2} \min_k |\lambda_k|^2 \sum_{k=1}^n |\eta_k|^2 \\ &\gtrsim \tau_2(z', \delta)^{-2} \|\eta\|^2, \end{aligned}$$

since $\tau_2(z', \delta)$ is the largest of the numbers τ_1, \dots, τ_n . This completes the proof of the proposition.

Remark. Although unimportant for our application of Proposition 3.2, in other contexts it is useful to exploit the fact that ϕ satisfies the stronger, non-isotropic estimate

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} (z) \xi_i \bar{\xi}_j \gtrsim \sum_{i=1}^n \left(\frac{|\xi_i|}{\tau_i(z', \delta)} \right)^2$$

for $z \in P_\delta(z')$ and $\xi \in \mathbb{C}^n$.

4. PROOF OF THEOREM 1.1

Let V be a complex analytic variety through p with order of contact m with $b\Omega$. For small $t > 0$, let $\{g^t\}$ be the family of complex manifolds associated to V given by Theorem 2.1. Without loss of generality, we may suppose that $g^t(0) \rightarrow p$ as $t \rightarrow 0$ and, perhaps after a translation, that $g^t(0) \in \Omega$ for small enough t . Let $z' = g^t(0)$ and apply Proposition 3.1 with $z' = z'$, for each z' , obtaining the coordinates (z'_1, \dots, z'_n) and the associated polydiscs. When working in any of the polydiscs, we will express all functions (such as the components of the parameterization of V) in terms of the associated coordinates. If we recall how the coordinate z_1 was constructed, it follows that we may assume that $\operatorname{Re} z_1$, oriented positively, points in the direction of $b\Omega \cap U$. Since $b\Omega$ is smooth, note that $(z'_1 \cdot z_1^s) / (\|z'_1\| \|z_1^s\|) \gtrsim 1$, independent of small t and s .

If $z \in U \cap \Omega$ and v is any unit vector in \mathbb{C}^n , let $\delta(z; v)$ denote the distance from z to $b\Omega$ along the complex line through v . An easy consequence of the convexity of Ω in U is that there is a constant C , independent of v and z , so that

$$\delta(z; v) v \in CP_\delta(z). \quad (4.1)$$

Here, as before, $\delta = \delta(z)$ and we are considering the vector v as based at z .

Now set $t = \delta^{1/m}$. Because of (iv) in Theorem 2.1, we can assume, without loss of generality, that $\delta(z') = t^m$, so, with respect to the new parameter δ ,

we have that $\delta(z') = \delta$, which then agrees with our previous notation. We claim that there is a constant c , independent of δ , so that

$$\text{vol}\{\zeta \in B(0, \delta^{1/m}) : g^{\delta^{1/m}}(\zeta) \subset P_\delta(z^{\delta^{1/m}})\} = c\delta^{2/m}. \quad (4.2)$$

In fact, if g'_1 does not identically vanish, it must map an open neighborhood of $0 \in \mathbb{C}$ onto a similar open neighborhood. Consider a sequence of points, $\{p_k\}$, which g'_1 maps to the real axis. The convexity of r shows that $|g'_1(p'_k)| \leq |r \circ g'(p'_k)|$. Thus, $v(g'_1) \geq v(r \circ g')$. Taylor's theorem then shows that the manifolds \tilde{M}_t , defined by $(0, g'_2(z), \dots, g'_n(z))$, have the same properties contained in Theorem 2.1 as the manifolds M_t . It is enough, therefore, if we show that (4.2) holds assuming $g'_1 \equiv 0$.

However, for any $\zeta \in B(0, \delta^{1/m})$, the real ray from $z^{\delta^{1/m}}$ (the origin in the (z'_1, \dots, z'_n) coordinates) to $(0, g^{\delta^{1/m}}_2(\zeta), \dots, g^{\delta^{1/m}}_n(\zeta))$ intersects $b\Omega \cap U$, by our coordinate construction. On the other hand, (iv) of Theorem 2.1 implies that

$$|r(g^{\delta^{1/m}}(\zeta))| \leq \delta, \quad \text{for all } \zeta \in B(0, \delta^{1/m}). \quad (4.3)$$

Therefore, (4.1) implies that

$$\{(0, g'_2(\zeta), \dots, g'_n(\zeta)) : \zeta \in B(0, \delta^{1/m})\} \subset CP_\delta(z^{\delta^{1/m}}),$$

which gives the modified (4.2).

For each δ , let ϕ^δ be the function given by Proposition 3.2. Applying Green's theorem to the functions $\phi^\delta \circ g^{\delta^{1/m}}$ gives

$$\begin{aligned} \int_{B(0, \delta^{1/m})} \Delta(\phi^\delta \circ g^{\delta^{1/m}}) dV(\zeta) &= \int_{bB(0, \delta^{1/m})} \partial_v(\phi^\delta \circ g^{\delta^{1/m}}) d\sigma(\zeta) \\ &\leq (C \cdot \delta^{-1/m}) \cdot 2\pi\delta^{1/m} \\ &\lesssim 1. \end{aligned} \quad (4.4)$$

Here we have used the convexity property of averages of subharmonic functions to obtain the first inequality. Computing the left-hand side of (4.4)—using (4.2), Proposition 3.2, (iii) of Theorem 2.1, and the fact that $\tau_2(q, \delta) \lesssim \delta^{1/L}$ for any $q \in U$ —we obtain

$$\begin{aligned} &\int_{B(0, \delta^{1/m})} \Delta(\phi^\delta \circ g^{\delta^{1/m}}) dV(\zeta) \\ &= \int_{B(0, \delta^{1/m})} \sum_{i,j=1}^n \frac{\partial^2 \phi^\delta}{\partial z_i \partial \bar{z}_j} (g^{\delta^{1/m}}(\zeta)) \frac{\partial g^{\delta^{1/m}}_i}{\partial \zeta} \frac{\overline{\partial g^{\delta^{1/m}}_j}}{\partial \bar{\zeta}} dV(\zeta) \\ &\gtrsim \tau_2(z^{\delta^{1/m}}, \delta)^{-2} \int_{B(0, \delta^{1/m})} |dg^{\delta^{1/m}}|^2 dV(\zeta) \\ &\gtrsim \tau_2(z^{\delta^{1/m}}, \delta)^{-2} \int_{B(0, \delta^{1/m})} dV(\zeta) \\ &\gtrsim \delta^{-(2/L) + (2/m)}. \end{aligned} \quad (4.5)$$

Thus, (4.4) and (4.5) show that $m \leq L$. Since $L \leq \Delta_1(0)$ from the definitions, the proof of Theorem 1.1 is complete.

Remark. Recently, Boas and Straube [B-S] have discovered a simpler, more geometric proof of Theorem 1.1.

REFERENCES

- [B-S] H. BOAS AND E. STRAUBE, On the equality of line type and variety type of real hypersurfaces in \mathbb{C}^n , preprint.
- [C] D. W. CATLIN, Necessary conditions for subellipticity of the $\bar{\partial}$ -Neumann problem, *Ann. of Math. (2)* **117** (1983), 147–171.
- [Ch] J.-H. CHEN, “Estimates of the Invariant Metrics on Convex Domains,” Ph.D. dissertation, Purdue University, 1989.
- [D’A] J. P. D’ANGELO, Real hypersurfaces, orders of contact, and applications, *Ann. of Math. (2)* **115** (1982), 615–637.
- [F] J. E. FORNÆSS, personal communication.
- [F-K] C. FEFFERMAN AND J. J. KOHN, Hölder estimates on domains of complex dimension two and on three dimensional CR manifolds, *Adv. Math.* **69** (1988), 233–303.
- [F-S] J. E. FORNÆSS AND N. SIBONY, Construction of P.S.H. functions on weakly pseudoconvex domains, *Duke Math. J.* **58** (1989), 633–655.